Point Vortex Dynamics on Curved Surfaces: A Hybrid Solver and a Hidden Jacobian Invariant

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We study point-vortex dynamics for incompressible flows constrained to closed, curved surfaces of arbitrary genus. Numerically, we develop a hybrid Lagrangian–Eulerian method: point vortices are treated as Lagrangian particles carrying vorticity, while the surrounding velocity field is recovered on a triangulated surface via discrete exterior calculus (DEC). This gives a sparse, vorticity-preserving representation that avoids geometry-specific Green/Robin functions and naturally incorporates the topology of the surface. Geometrically, we make the handle circulations explicit via harmonic 1-forms and the Riemann-surface structure, and show that vortex positions together with these global circulation variables define a conserved point in the Jacobian torus of the surface (a canonical configuration space encoding how flows wind around the surface's handles). Via the standard correspondence between the Jacobian variety and the Picard group, this can be interpreted as a nontrivial invariant in complex analysis and algebraic topology: the isomorphism class of a certain holomorphic line bundle associated with the vortex configuration is preserved by the Euler flow. This brief note focuses on the high-level structure, numerical formulation, and current experiments; full derivations and proofs are contained in an extended report.

Additional Key Words and Phrases: point vortices, incompressible flow, discrete exterior calculus, Riemann surfaces, Jacobian, geometric invariants

1 Big Picture

Imagine a handful of tiny whirlpools sliding around on a curved surface like a sphere, a torus, or something more complicated. The fluid always stays tangent to the surface; the vortices move, interact, and push each other around.

In the plane, this picture is classical: point vortices give a clean Hamiltonian system, and complex analysis provides closed-form formulas for the velocity field. This model is attractive because it is sparse (all vorticity lives at a few moving points), vorticity-preserving (no numerical diffusion from smearing vorticity on a grid), and rich enough to capture many phenomena of two-dimensional Euler flow.

On a curved, closed surface, two new features appear:

- **Geometry.** Curvature modifies how vortices drift and interact; even a vortex configuration that would be stationary in the plane can move nontrivially.
- **Topology.** On a surface with handles, fluid can circulate around non-contractible loops indefinitely, even in regions with no local vorticity.

Analytically, there are beautiful works extending point-vortex dynamics to special geometries using Green and Robin functions, but these approaches are typically tied to highly symmetric surfaces and are difficult to transport directly to arbitrary triangulated meshes. Naive numerical approaches, on the other hand, often treat the surface almost like a deformed planar patch and effectively ignore the extra topological degrees of freedom.

The core question motivating this project is:

Can we build a numerical method for point vortices on arbitrary closed surfaces that is practical on real meshes, explicitly aware of topology, and still faithful to the structure of the analytical theory?

In trying to answer this question numerically, we also stumble on a surprising geometric feature: when we track vortex positions together with the right global circulation variables, their combined state turns out to define a conserved point in a special torus associated to the surface, namely its Jacobian variety.

This note outlines my current answer: a hybrid solver based on discrete exterior calculus, together with a geometric viewpoint that packages vortex data and handle circulations into a conserved point in the Jacobian torus of the surface, a natural torus-shaped configuration space that records how flows wind around the handles.

Key ideas at a glance

- **Hybrid solver.** Point vortices as Lagrangian particles, plus a mesh-based Poisson solve on a triangulated surface using DEC.
- **Topology made explicit.** The solver tracks how fluid circulates around each handle via harmonic 1-forms, rather than hiding this in geometry-specific Green/Robin functions.
- **Hidden invariant.** Vortex positions and handle circulations together define a conserved point in a torus (the Jacobian variety of the associated Riemann surface). This gives a geometric version of Kelvin's circulation theorem and, via Riemann-surface and algebraic-geometry language, is equivalent to preserving the isomorphism class of a certain holomorphic line bundle.

Full derivations, proofs, and implementation details are given in an extended technical report; here I focus on the conceptual structure and why this project is exciting as a platform for structure-preserving simulation and as an example of a nontrivial invariant bridging fluid dynamics and complex geometry.

2 A Hybrid Lagrangian–Eulerian Solver on a Surface

Let M be a closed orientable surface with a triangle mesh discretization. We model incompressible Euler flow on M with vorticity concentrated at point vortices: the continuous vorticity 2-form is

$$\omega = \sum_i \Gamma_i \, \delta_{x_i}(t),$$

where Γ_i are vortex strengths and $x_i(t)$ are their positions.

The numerical method is a hybrid Lagrangian–Eulerian scheme:

- **Lagrangian side.** Vortices are particles carrying strengths Γ_i at positions $x_i(t)$. They move with the local fluid velocity, keeping vorticity sharp and localized.
- **Eulerian side.** The surrounding velocity field is reconstructed on the surface mesh by solving a Poisson equation for a stream function.

Extended technical report : https://rudresh.net/assets/pdf/PointVortexDynamics.pdf.

In the classical stream-function formulation of 2D incompressible flow, one introduces a scalar function ψ such that the velocity field is obtained by a 90-degree rotation of its gradient, $v = J\nabla\psi$. This automatically enforces incompressibility, and the vorticity is given by $\omega = \Delta\psi$. Thus, given vorticity, one recovers the velocity by solving a Poisson equation for ψ. Our surface formulation mirrors this idea in a geometric setting.

On a simply connected planar domain, this stream-function description is essentially complete: every divergence-free velocity field can be written, up to a gradient, in terms of a single-valued scalar ψ. On higher-genus surfaces, however, there exist divergence-free flows with no globally defined stream function at all. These purely topological "handle-circulation" modes are exactly the extra global degrees of freedom that we make explicit in the next section.

DEC provides discrete analogues of gradients, curls, and Hodge stars on meshes, so that the usual 2D incompressible-flow relations are preserved in a combinatorial form. For the present overview, the key behaviors are:

- The method avoids explicit Green/Robin functions for each new geometry; the mesh and DEC take their place.
- Vorticity remains concentrated in point vortices, yielding a sparse, vorticity-preserving representation even on comparatively coarse meshes.

In effect, we retain the classical intuition of "vortex spots pushing each other around" while using a geometry-aware mesh discretization to handle arbitrary surfaces.

3 Topology and a Torus-Shaped Invariant

On a simply connected domain, an incompressible velocity field is essentially determined by its vorticity via a single scalar stream function: up to a gradient, every divergence-free field can be written as a rotated gradient of ψ. On a surface with handles, this is no longer true: there are additional global degrees of freedom corresponding to flows that wrap around non-contractible loops and cannot be represented by any single-valued stream function.

A geometric way to see this uses Hodge decomposition. Let $u = v^\flat$ be the velocity 1-form. On a closed Riemannian surface, Hodge theory tells us that we can decompose

$$u = d\phi + \star d\psi + h,$$

where $d\phi$ is an exact (gradient) part, $\star d\psi$ is the stream-function part determined by vorticity, and h is a *harmonic* 1-form, simultaneously closed and co-closed. For incompressible flows we quotient out gradients, leaving a divergence-free part of the form

$$u = \star d\psi + h.$$

Hodge's theorem further implies that the space of harmonic 1-forms on a genus-g surface has real dimension $2g$. One can choose a basis of loops $\{\gamma_k\}_{k=1}^{2g}$ spanning $H_1(M;\mathbb{Z})$ and a dual basis of harmonic 1-forms $\{h_k\}_{k=1}^{2g}$.

Recent work in fluid cohomology shows that these handle circulations evolve in time and are coupled to the motion of the vortices via Kelvin's circulation theorem. In terms of the decomposition above, the vortex configuration determines the vorticity and hence the stream function ψ via the Poisson solve, while the harmonic part can be written as

$$h(t) = \sum_{k=1}^{2g} c_k(t)\, h_k,$$

where the coefficients $c_k(t)$ encode exactly these global circulation degrees of freedom.

In my formulation, the state of the flow therefore consists of

- the list of vortex positions and strengths (which fix ω and hence ψ), and
- a vector $c(t) \in \mathbb{R}^{2g}$ encoding handle circulations (equivalently, coefficients in a harmonic 1-form basis).

From the continuous Kelvin circulation theorem one can derive an ODE for these coefficients $c_k(t)$. In the complex-analytic picture below, this evolution is exactly what ensures that a suitable Abel–Jacobi coordinate of the flow remains constant.

This picture has a natural home in complex analysis. Every closed orientable surface admits a compatible complex structure, and any smooth Riemannian metric determines a conformal class. It is thus natural to regard M as a compact Riemann surface. In that setting:

- The vortex positions and strengths define a degree-zero *divisor*
$$D(t) = \sum_i \Gamma_i\, x_i(t),$$
which is just a formal signed sum of points on the surface with weights given by the vortex strengths. The classical Abel–Jacobi map then sends such a divisor to a point $\mathrm{AJ}(D(t))$ in the Jacobian variety $\mathrm{Jac}(M)$, a $2g$-dimensional real torus built from periods of holomorphic 1-forms. Intuitively, $\mathrm{AJ}(D(t))$ measures, via path integrals of holomorphic 1-forms, how the vortex configuration "winds" around the independent cycles of the surface.
- The harmonic 1-form coefficients $c(t)$, or equivalently their periods along the homology basis, determine another point in the same torus via the standard identification of $\mathrm{Jac}(M)$ with the space of harmonic cohomology classes modulo the integer lattice.

A central observation of this project is that one can combine these two pieces into a *single Jacobian-valued invariant*. More concretely, we construct a map

$$J\colon (D(t), c(t)) \longmapsto \mathrm{Jac}(M), \qquad J(D, c) = \mathrm{AJ}(D) + H(c),$$

for a suitable linear map H sending the harmonic coordinates to the Jacobian, and show at the differential-form level that

$$\frac{d}{dt} J(D(t), c(t)) = 0$$

along the point-vortex evolution. Informally:

As the vortices slide around and the handle circulations adjust, their combined effect traces out a constant point in the Jacobian variety, a torus-shaped configuration space attached to the surface.

This viewpoint gives a geometric way to see Kelvin's circulation theorem: not just as a family of integrals along moving loops, but as a fixed point in an intrinsic configuration space that the dynamics cannot leave. In the extended report, the point-vortex equations of motion are written in terms of period integrals of meromorphic 1-forms, making the Abel–Jacobi map appear explicitly. Through the classical identification $\mathrm{Pic}^0(M) \cong \mathrm{Jac}(M)$, this invariant can be rephrased as follows: the isomorphism class of a certain holomorphic line bundle, whose divisor encodes the vortices and whose flat unitary connection encodes the handle circulations, is preserved by the (generally non-conformal) fluid flow.

4 Implementation and Current Experiments

I have implemented this solver on arbitrary closed triangle meshes in a small framework I am building, tentatively called *Rheidos*. At a high level, the solver pipeline on a given mesh proceeds as follows.

Precomputation. On the fixed mesh, DEC operators are assembled:

- primal/dual incidence matrices and Hodge star operators for 0-, 1-, and 2-cochains,
- the scalar Laplace–Beltrami operator for the Poisson solve, and
- an L^2-orthonormal basis $\{\eta_k\}_{k=1}^{2g}$ of discrete harmonic 1-forms, computed via a standard tree–cotree construction followed by projection to the harmonic subspace.

Time stepping. Given a vortex configuration $\{x_i(t), \Gamma_i\}$ and harmonic coefficients $c_k(t)$, a single time step consists of:

(1) Depositing the point-vortex vorticity to a discrete 2-cochain on faces using barycentric weights.
(2) Solving the discrete Poisson equation for the stream function ψ with a gauge constraint to fix the additive constant.
(3) Forming the discrete velocity 1-form

$$u = \star d\psi + \sum_{k=1}^{2g} c_k(t)\, \eta_k,$$

and converting this to a piecewise-linear tangent vector field on the surface.
(4) Evaluating this vector field at each vortex position (via barycentric interpolation in its containing triangle) to obtain $\dot{x}_i$, and advancing the positions with a standard ODE integrator (e.g., explicit Runge–Kutta).
(5) Updating the harmonic coefficients $c_k(t)$ using an ODE derived from the discrete Kelvin constraints, so that the discrete handle circulations evolve consistently with the vortex motion.

On genus-one examples (torus meshes), the implementation already shows:

- vortex configurations whose collective motion exhibits curvature-induced drift consistent with analytical expectations where available,
- nontrivial interactions of multiple vortices constrained to the surface, and
- a discrete Jacobian-valued quantity $J_h(t)$ that remains nearly constant along simulated trajectories. The observed drift decreases with mesh refinement and smaller time steps, reflecting the fact that our current time integrator and Poisson solve are not exactly structure-preserving but approximate the continuous invariant to high accuracy.

The extended report contains the precise DEC constructions and the formal derivation of the invariant. For this brief note, the main message is that the project is not just conceptual: there is running code, visual demos, and a clear path to more systematic experiments.

5 Conclusion

This work is an example of how nontrivial invariants can emerge when one combines discrete differential geometry, complex analysis, and fluid dynamics: a flow that is not conformal still preserves a holomorphic line-bundle class. I am interested in using this kind of structure in future projects, both to guide the design of fast, robust physics solvers and, where appropriate, by leveraging physics-informed or geometry-aware learning methods by encoding such invariants as constraints rather than ignoring them.